Functions of Two or More Independent Variables

INTRODUCTION

The values of many real-world functions are determined by more than one independent variable. For example, the function $V = \pi r^2 h$ calculates the volume of a circular cylinder from its radius and height, the area function of a rectangular shape depends on both its width and its height. And, the pressure of a given quantity of gas varies with respect to the temperature of the gas and its volume. We define a function of two variables as follows:

A function f of two variables is a relation that assigns to every ordered pair of input values (x, y) in a set called ***the domain*** *a unique output value denoted by f (x, y) . The set of output values is called* ***the range.***

PARTIAL DERIVATIVE

Let $f(x, y)$ be defined in a region R of the xy-plane. If we think of y as fixed and x as variable, the derivative of $f(x, y)$ with respect to x is called the *partial derivative* with respect to x. This partial derivative is denoted by $\frac{df}{dx}$, if we write u = $f(x, y)$, the partial derivative is also denoted by $\frac{du}{dx}$. Likewise, the partial derivative with respect to y, is $\frac{df}{dy}$ or $\frac{du}{dy}$, is the derivative of $f(x, y)$ with respect to y when x is regarded as a constant.

Main Ref. 1. E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS".
2. L. Finny & B. Thomas, "CALCULAS II".

Example: - If u is function of (x, y) as:-

$$u(x, y) = x^2 y + e^{-xy^3}$$

$\frac{du}{dx}$

$\frac{du}{dy}$

Similar definitions and notations apply in dealing with functions of three or more independent variables.

Hence, the partial derivatives of $\frac{d^2 f}{dx^2}$ and $\frac{d^2 f}{dy^2}$ are called the second partial derivatives. There are in all four second derivatives of $f(x, y)$. The notations for these derivatives, if we write u = $f(x, y)$, are the following:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \quad , \quad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\, \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\, \partial x} \quad , \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

Example: - For the given function

$u(x, y) = x^2 y + e^{-xy^3}$ then:-

CHAIN RULE

Let w = f (x, y, z) be continuous and have continuous first partial derivatives in a domain D in xyz-space. Let x: x (u, v), y: y (u, v), z: z (u, v) be functions that are continuous and have first partial derivatives in a domain B in the uv-plane, where B is such that for every point (u, v) in B, the corresponding point [x(u, v), y(u, v), z(u, v)] lies in D. See Fig. 213. Then the function

$$w = f\big(x(u, v), y(u, v), z(u, v)\big)$$

then

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$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Also,

$$w=f\big(x(u),y(u),z(u)\big)$$

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{dx}{du}+\frac{\partial w}{\partial y}\frac{dy}{du}+\frac{\partial w}{\partial z}\frac{dz}{du}$$

Differential operator (∇) (***Dell***)

Certain differential operations may be performed on scalar and vector fields and have wide ranging applications in the physical sciences. The most important operations are those of finding the gradient of a scalar field and the divergence and curl of a vector field. It is usual to define these operators from a strictly mathematical point of view, as we do below. In the following sections, however, we will discuss their geometrical definitions, which rely on the concept of integrating vector quantities along lines and over surfaces. Central to all these differential operations is the vector operator ∇, which is called **Del** (or sometimes **nabla**) and in Cartesian coordinates is defined by:-

$$\nabla=\frac{\partial}{\partial x}\,i+\frac{\partial}{\partial y}\,j+\frac{\partial}{\partial z}\,k$$

GRADIENT

Scalar function $f(x,y,z)$ that is defined and differentiable in a domain in 3-space with Cartesian coordinates *x, y, z*. We denote the **gradient** of that function by (grad *f*) or ∇f . Then the gradient of is defined as the vector function :-

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Algebraic Rules for Gradient

1. Constant Multiple Rule: $\nabla(kf) = k\nabla f \quad k \text{ is any constant}$
2. Sum Rule: $\nabla(f + g) = \nabla f + \nabla g$
3. Difference Rule: $\nabla(f - g) = \nabla f - \nabla g$
4. Product Rule: $\nabla(f * g) = g * \nabla f + f * \nabla g$

Directional Derivative

Direction derivative of the surface $\varphi\ (x, y, z) = 0$ in to the direction of a given vector $\vec{B}$ is: $D.D.f = \nabla\varphi \,.\, \overrightarrow{U_B}$

Theorem

The maximum value of the directional Derivatives of $f\ (x, y, z)$ is $|\nabla f|$

DIVERGENCE

If the vector $\vec{v}$ be a differentiable vector function, in Cartesian coordinates, and let (v_1, v_2 and v_3) be the components of $\vec{v}$. Then the function

$$div\ \vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Is called the divergence of $\vec{v}$ or the ***divergence of the vector*** field defined by v. Another common notation for the divergence is

$$div\ \vec{V} = \nabla \cdot \vec{V} = \left(\frac{\partial}{\partial x}\ i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (v_1\ i + v_2\, j + v_3\, k)$$

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$$div\,\vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \quad \textit{is the } \textbf{\textit{laplasian equation}}$$

CURL

Let $\vec{v}$ be a differentiable vector function of the Cartesian coordinates x, y, z. and let (v1, v2 and v3) be the components of $\vec{v}$. Then the curl of the vector function $\vec{v}$ or of the vector field given by $\vec{v}$ is defined by:

$$Curl\,\vec{V} = \nabla \times \vec{V} = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Always Remember that:

Curl grad $f = 0$

Or $\quad \nabla \times \nabla f = 0$

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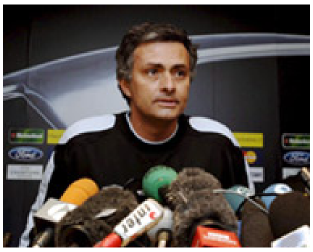
Vector Identity

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}\equiv(\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}\equiv(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\equiv(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$
$$\nabla\cdot(\mathbf{A}+\mathbf{B})\equiv\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}$$
$$\nabla(V+W)\equiv\nabla V+\nabla W$$
$$\nabla\times(\mathbf{A}+\mathbf{B})\equiv\nabla\times\mathbf{A}+\nabla\times\mathbf{B}$$
$$\nabla\cdot(V\mathbf{A})\equiv\mathbf{A}\cdot\nabla V+V\nabla\cdot\mathbf{A}$$
$$\nabla(VW)\equiv V\nabla W+W\nabla V$$
$$\nabla\times(V\mathbf{A})\equiv\nabla V\times\mathbf{A}+V\nabla\times\mathbf{A}$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B})\equiv\mathbf{B}\cdot\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\mathbf{B}$$
$$\nabla(\mathbf{A}\cdot\mathbf{B})\equiv(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})$$
$$\nabla\times(\mathbf{A}\times\mathbf{B})\equiv\mathbf{A}\nabla\cdot\mathbf{B}-\mathbf{B}\nabla\cdot\mathbf{A}+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$$
$$\nabla\cdot\nabla V\equiv\nabla^2 V$$
$$\nabla\cdot\nabla\times\mathbf{A}\equiv 0$$
$$\nabla\times\nabla V\equiv 0$$
$$\nabla\times\nabla\times\mathbf{A}\equiv\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

Any Question Please



Excuse me...what is this again??!!!

GRADIENT

CARTESIAN	$\nabla V=\dfrac{\partial V}{\partial x}\mathbf{a}_x+\dfrac{\partial V}{\partial y}\mathbf{a}_y+\dfrac{\partial V}{\partial z}\mathbf{a}_z$
CYLINDRICAL	$\nabla V=\dfrac{\partial V}{\partial \rho}\mathbf{a}_\rho+\dfrac{1}{\rho}\dfrac{\partial V}{\partial \phi}\mathbf{a}_\phi+\dfrac{\partial V}{\partial z}\mathbf{a}_z$
SPHERICAL	$\nabla V=\dfrac{\partial V}{\partial r}\mathbf{a}_r+\dfrac{1}{r}\dfrac{\partial V}{\partial \theta}\mathbf{a}_\theta+\dfrac{1}{r\sin\theta}\dfrac{\partial V}{\partial \phi}\mathbf{a}_\phi$

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DIVERGENCE

CARTESIAN
$$\nabla \cdot \mathrm{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

CYLINDRICAL
$$\nabla \cdot \mathrm{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

SPHERICAL
$$\nabla \cdot \mathrm{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

CURL

CARTESIAN
$$\nabla \times \mathrm{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

CYLINDRICAL
$$\nabla \times \mathrm{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right]\mathbf{a}_z$$

SPHERICAL
$$\nabla \times \mathrm{H} = \frac{1}{r\sin\theta}\left[\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right]\mathbf{a}_\phi$$

LAPLACIAN

CARTESIAN
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL
$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL
$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

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